Quant Corner $C = SN(d_+) - Ke^{-r\tau}N(d_-)$

Formulating a Revolution

Myron Scholes discusses the creation of the options pricing equation that won him a Nobel prize and ignited the $236 trillion derivatives market.

By Peter Carr

WHEN MYRON SCHOLES and Fischer Black first submitted their paper on a new method of valuing stock options to *The Journal of Political Economy* in 1970 and then to *The Review of Economics and Statistics* in 1971, the journals rejected it. *The Journal of Political Economy* sent it to a referee for review, who decided it was too arcane, and *The Review of Economics and Statistics* turned it down without review, Scholes says. The journals rejected a study, "The Pricing of Options and Corporate Liabilities," that would revolutionize finance and win Scholes and Robert Merton, who later refined the model, the Nobel Memorial Prize in Economic Sciences in 1997.

Eventually published in *The Journal of Political Economy* in 1973, the Black-Scholes formula gave a boost to the derivatives industry by making it possible for the first time to price and hedge stock option positions with precision. Similar formulas were soon devised by Mark Garman, Steven Kohlhagen and other academics for currency and fixed-income options and a dizzying array of derivatives of increasing complexity. Garman and Kohlhagen developed their formula for valuing currency options while both were professors at the University of California, Berkeley, in the early 1980s. Today, one can invest in financial instruments with such esoteric names as correlation swaps, Napoleons, reverse cliquets, risk reversals and volatility bonds.

The Black-Scholes formula does have its detractors, including even Fischer Black, the co-author of the article, who died in 1995. In an article titled "The Holes in Black-Scholes," published in *Risk* in 1992, Black said the formula doesn't recognize that volatility is random and that volatility usually has a negative correlation with the returns of the underlying stock. Today, hundreds of universities around the world offer postgraduate courses in mathematical finance. These courses typically teach the mathematics that led to the formula and then show how those same techniques can be used to improve on the original concept. Thousands of articles have been published showing how the formula must be tweaked before it can be applied to current products.

Scholes himself says the Black-Scholes formula has required some refinement through empirical studies of actual data to enable its current applications in derivatives pricing models. "Every model is an incomplete description of reality, and we need to learn from both experience and theory," says Scholes, 64, who's now a professor emeritus of finance at Stanford University in Stanford, California. He's also chairman of Oak Hill Platinum Partners LLC, a Rye Brook, New York–based hedge fund firm that manages $2 billion.

Myron Scholes

The Black–Scholes formula for valuing options

Current job: Chairman of Oak Hill Platinum Partners in Rye Brook, New York, and professor emeritus of finance at Stanford University

Background: Earned a bachelor's degree from McMaster University in Hamilton, Ontario. Received an MBA and a Ph.D. in finance from the University of Chicago. Has been a professor of finance at various levels at MIT's Sloan School of Management in Cambridge, Massachusetts; the University of Chicago Graduate School of Business; and the Stanford University Graduate School of Business in Stanford, California. With Robert Merton, received the Nobel Memorial Prize in Economic Sciences in 1997.

Personal: Age 64. Married.

IN 1970, SCHOLES, who had just earned a Ph.D. from the University of Chicago, was an assistant professor of finance at the Sloan School of Management at the Massachusetts Institute of Technology in Cambridge, Massachusetts. He was looking for a compelling research topic and a colleague at MIT introduced him to Black, a consultant at Arthur D. Little, who had a Ph.D. in math from Harvard University and a penchant for exploring applications of the capital asset pricing model for valuing securities. Scholes met Black for lunch at Arthur D. Little's office in Cambridge. "He was interested in a lot of ideas in finance, and I was a young assistant professor with thousands of ideas in my head," says Scholes, who was in his late 20s at the time. "And so we just had the start of a terrific discussion."

At the time, the CAPM was the standard academic approach for pricing any financial asset. That formula had been developed in the previous decade by Black's mentor, Jack Treynor—who's

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now president of Treynor Capital Management Inc.—and three other researchers. As traditionally formulated, the CAPM applies to a timeless world in which all asset returns are normally distributed, taking the form of a bell-shaped curve. The CAPM assumes that investors know the mean return vector (the expected path of returns over time) and the covariance matrix (which quantifies how the assets in the portfolio affect each other's returns). The formula also assumes that investors agree with each other on their elements. The central lesson of the CAPM is that risk is priced in as the security's beta, defined as the slope in a regression of the security's return on the market portfolio's return. Beta measures the sensitivity of the security's price to the return of the market.

Together, Black and Scholes attempted to apply the CAPM to the pricing of stock options. With their skewed payoffs, puts and calls seem to fall outside the purview of the assumptions underlying the traditional CAPM. A put is an option that gives the holder the right to sell the underlying shares at a fixed price and at a fixed time. Likewise, a call option gives the holder the right to buy the underlying security. The slopes of both types of options tend to change abruptly when the underlying stock reaches the option's exercise price, especially near the expiration date.

Fortunately, Robert Merton, another MIT professor, had just reformulated the CAPM to apply in continuous time. Continuous-time models provide a more realistic valuation than discrete-time models, which impose a minimum trading interval. Black and Scholes theorized that the returns on puts and calls are normally distributed if these returns are viewed over a sufficiently short horizon. If they could determine the option's beta, then the CAPM would provide the pricing rule. The problem is that if the stock's beta is constant over time, as is usually assumed, then the option's beta varies with time.

Any finance professor at the time would have recognized this problem. What distinguished Black and Scholes was their realization that the CAPM could be applied to a portfolio of securities just as easily as it could be applied to individual stocks.

BLACK AND SCHOLES realized that if they could form a portfolio of the option and its underlying stock that had zero beta, then the CAPM would guarantee that such a portfolio would earn the risk-free rate of return in equilibrium. That's because the CAPM states that an asset's expected return is the risk-free rate of return plus its beta multiplied by the expected excess return on the market portfolio. By forming a zero-beta portfolio, Black and Scholes could price it. And since they assumed the price of the underlying stock was directly observable, they could indirectly determine the option price. Because they also assumed that the beta of the stock was constant and the beta of the option varied with time, it became clear to Black and Scholes that a portfolio whose beta is constant at zero requires continuous rebalancing of the position in either or both assets.

Merton had also developed the theory of continuous portfolio rebalancing, often called dynamic trading, which enabled Black and Scholes to form a portfolio with zero beta. As it relies heavily on an esoteric branch of probability theory called Itô calculus, we won't pursue the details here. Suffice it to say that Black and Scholes showed that with dynamic trading in the option and its underlying stock, the portfolio of these two assets can be made to have zero beta at all times, as shown in figure 1. They invoked Merton's continuous-time CAPM to conclude they could price the portfolio. The resulting determination of the option price involved solving a complicated partial differential equation, whose solution stumped the two financial economists for months.

Here's the formula that Black and Scholes developed:

$$C = SN(d_+) - Ke^{-r\tau}N(d_-),$$

$$N(d) \equiv \int_{-\infty}^{d} \frac{e^{-z^2/2}}{\sqrt{2\pi}}dz,$$

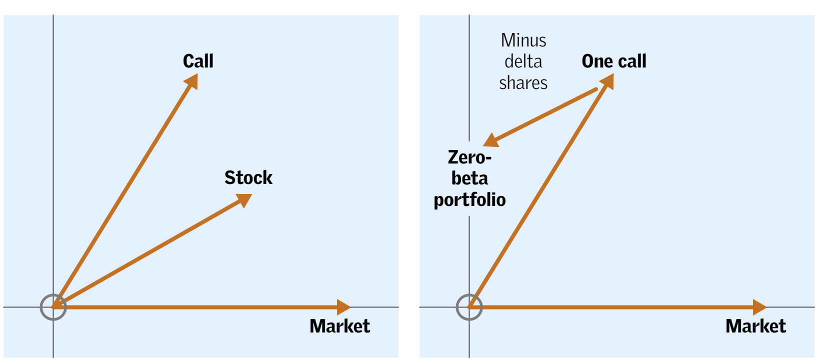
$$d_\pm = \frac{\ln(S/K)+(r\pm\sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

Here:

- C is the call value to be determined.
- S is the known stock price.
- K is the strike price, the fixed amount specified in the option's contract to be paid at maturity if the call is exercised.
- r is the interest rate for both borrowing and lending, assumed constant over the call's lifetime τ.
- $N(d)$ denotes the normal, or Gaussian, distribution function, the probability

Figure 1: The imperfect correlation case

The length of each vector represents the standard deviation of each asset's return, and the angle of each vector is related to its beta.



Market = The return of the market portfolio.
Delta = The sensitivity of the call's price to a small change in the value of the underlying stock.

Source: Bloomberg

that a standardized Gaussian random variable realizes below d.

- d_+ and d_- are alternative measures of the moneyness of the call, indicating by how many standard deviations $\sigma\sqrt{\tau}$ the concavity-corrected return $\ln(S_T/S_0)$ exceeds $\ln(Ke^{-r\tau}/S_0)$.

The key input to the formula is the stock's volatility σ, a positive constant that measures the variability in stock returns over time.

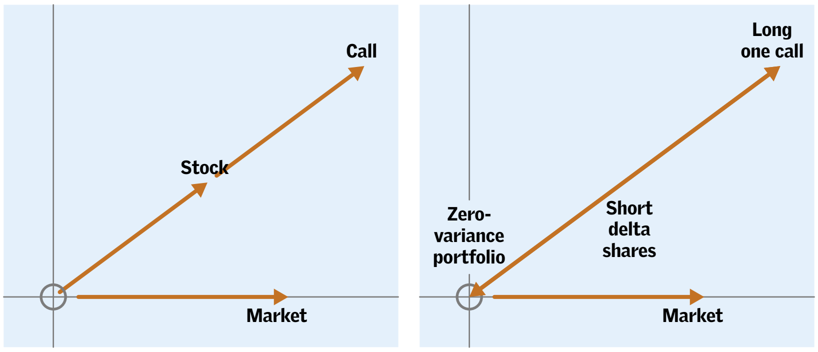
Just as important is what the call price doesn't depend on. Despite its genesis, the call price doesn't require most of the inputs to the continuous-time CAPM under which the formula was originally derived. The only element of the covariance matrix of asset returns that needs to be known is the underlying stock's return variance. Furthermore, the whole vector of mean returns doesn't need to be known. In particular, one doesn't require knowledge of the expected rate of return of the underlying stock, which is surprising at first glance. The invariance to mean returns is particularly important since estimating expected returns from time series is notoriously difficult.

The various terms making up the Black-Scholes formula all have interesting financial interpretations. $N(d_+)$ is called the stock's delta, since it indicates the increase in the call's model value for a small increase in the underlying stock price. As pointed out by Black and Scholes, delta is also the number of shares held in order to zero out the portfolio beta when short one call. $Ke^{-r\tau}$ is clearly the present value of the strike price; since the strike price is paid only at expiry, it makes sense that it must be present valued before it appears in today's value. Finally, $N(d_-)$ is now called the risk-neutral probability that the call finishes in the money, but this interpretation came after Black-Scholes.

Soon after Black and Scholes circulated their results, their colleague Merton showed that the same strategy that zeroes out portfolio beta zeroes out portfolio variance as well, as shown in figure 2. This observation allowed Merton to derive the Black-Scholes formula without using some of the assumptions of the continuous-time CAPM. In particular, Merton showed that it isn't necessary for investors to agree on estimates of expected return or to know the stock's beta. Black and Scholes liked Merton's derivation of their formula so much that they made it the first derivation that appears in their paper, crediting him in footnote 3. They then went on to impose the stronger assumptions of the continuous-time CAPM and used it to derive their formula again. Even today, Scholes says, "Merton's was a very elegant derivation."

Figure 2: The perfect correlation case

The zero-beta portfolio is the null vector and therefore has zero variance.



Market = The return of the market portfolio.
Delta = The sensitivity of the call's price to a small change in the value of the underlying stock.

Source: Bloomberg

THE MERTON DERIVATION is based on no arbitrage rather than equilibrium and is now the standard argument used to justify the Black-Scholes formula. No arbitrage is a less stringent assumption because a system in equilibrium has no arbitrage, while a system with no arbitrage may or may not be in equilibrium. However, Black was on record for favoring their original CAPM-based approach, and Scholes says he prefers the CAPM derivation.

One explanation is that the continuous-time CAPM continues to provide a unique price for an option when some of the original assumptions that Merton made are relaxed. For example, suppose, in contrast to Merton's assumptions, that one can't trade the underlying stock continuously, perhaps because it's very illiquid. However, suppose that one can continuously trade some other stock that doesn't underlie the option. So long as the liquid stock has nonzero beta, then one can form a zero-beta portfolio consisting of the option on illiquid stock A and liquid stock B.

To take another example, suppose that one can trade the underlying stock continuously but its volatility is random and imperfectly correlated with returns, as Black later argued. Then it's well known that it becomes impossible to eliminate variance in a portfolio consisting only of the option and its underlying stock. However, it's still possible to eliminate the beta of the portfolio using just these two assets.

Scholes says derivatives in general have become exponentially more complex than when he helped develop the Black-Scholes formula 33 years ago. "We are learning more and more about how to use derivatives to transfer risk and to hedge risks that we need not take," he says. Scholes also believes there's another possible result of this learning process. "As individuals learn, they are willing to take on additional risks, and, as a result, the possibilities for mishaps again increase," he says. Even taking into account the potential for problems, Scholes is still a believer in the derivatives industry that his work helped develop.

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